Published by Institute of Physics Publishing for SISSA
Received: *April 21, 2005*
Revised: *April 29, 2005*
Accepted: *April 29, 2005*
Published: *May 25, 2005*

Charges of twisted branes: the exceptional cases

Stefan Fredenhagen, Matthias R. Gaberdiel and Thomas Mettler

Institut für Theoretische Physik, ETH-Hönggerberg
CH–8093 Zürich, Switzerland
E-mail: stefan@itp.phys.ethz.ch, gaberdiel@itp.phys.ethz.ch,
thomas.mettler@student.ethz.ch

Abstract: The charges of the twisted D-branes for the two exceptional cases (SO(8) with the triality automorphism and E_6 with charge conjugation) are determined. To this end the corresponding NIM-reps are expressed in terms of the fusion rules of the invariant subalgebras. As expected the charge groups are found to agree with those characterising the untwisted branes.

Keywords: Conformal Field Models in String Theory, D-branes.

JHEP05(2005)058

Contents

1. Introduction

A lot of information about the dynamics of D-branes is encoded in their charges. In particular, the D-brane charges constrain possible decay processes, and thus play an important role in stability considerations. There is evidence that these charges take values in (twisted) K-theory [1−3]. For D-branes on a simply connected group manifold G, the charge group is conjectured to be the twisted K-theory ${}^{k+h^\vee}K(G)$ [4, 5], where the twist involves an element of the third cohomology group $H^3(G,\mathbb{Z})$, the Wess-Zumino form of the underlying Wess-Zumino-Witten (WZW) model at level k.

For all simple, simply connected Lie groups G, the twisted K-theory has been computed in [6] (see also [7, 8]) to be

$$ {}^{k+h^\vee}K(G) = \underbrace{\mathbb{Z}_{M(G,k)} \oplus \cdots \oplus \mathbb{Z}_{M(G,k)}}_{2^{\mathrm{rk}(G)-1}} , \tag{1.1} $$

where $M(G,k)$ is the integer

$$ M(G,k) = \frac{k+h^\vee}{\gcd(k+h^\vee,L)} . \tag{1.2} $$

Here $h^\vee$ is the dual Coxeter number of the finite dimensional Lie algebra $\bar{\mathfrak{g}}$, and L only depends on G (but not on k). In fact, except for the case of C_n that will not concern us in this paper, L is

$$ L = \mathrm{lcm}\{1,2,\ldots,h-1\} , \tag{1.3} $$

where h is the Coxeter number of $\bar{\mathfrak{g}}$. The above formula for $M(G,k)$ was obtained by reducing the K-theory calculation to a calculation in conformal field theory that had been previously done. For $\bar{\mathfrak{g}} = A_n$ the formula (1.2) was derived in [9] (see also [10]), while the formulae in the other cases were checked numerically up to very high levels in [11]. For the classical Lie algebras and G_2 an alternative expression for $M(G,k)$ was also derived in [8].

These results should be compared with the charges that can be determined directly in terms of the underlying conformal field theory. The idea behind this approach is that brane configurations that are connected by RG flows should carry the same charge. These constraints were used in [12] to determine the charge group of su(2). The constraint equations were generalised in [9] to the branes $a \in \mathcal{B}_k^\omega$ of an arbitrary WZW model that preserve the full affine symmetry algebra $\mathfrak{g}$ up to some automorphism ω. There it was argued that the charges q_a satisfy

$$\dim(\lambda)\; q_a = \sum_{b \in \mathcal{B}_k^\omega} \mathcal{N}_{\lambda a}{}^b \; q_b \qquad \mod M\,, \tag{1.4}$$

where $M \in \mathbb{N}$, $\lambda \in \mathcal{P}_k^+(\bar{\mathfrak{g}})$ is a dominant highest-weight representation of the affine Lie algebra $\mathfrak{g}$ at level k, $\dim(\lambda)$ is the Weyl-dimension of the corresponding representation of the horizontal subalgebra $\bar{\mathfrak{g}}$, and $\mathcal{N}_{\lambda a}{}^b$ are the NIM-rep coefficients appearing in the Cardy analysis. In this paper we shall ignore the low level ($k = 1, 2$) subtleties discussed in [11] and assume that k is sufficiently big ($k \geq 3$).

For the trivial automorphism ($\omega = \mathrm{id}$), the branes can be labelled by dominant highest weights of $\mathfrak{g}$, $\mathcal{B}_k^{\mathrm{id}} \cong \mathcal{P}_k^+(\bar{\mathfrak{g}})$. In this case, the constraints (1.4) were evaluated in [9, 11]. The charges are given (up to rescalings) by the Weyl-dimensions of the corresponding representations, $q_\lambda = \dim(\lambda)$, and the charge is conserved only modulo $M = M(G,k)$. Thus, the untwisted branes account for one summand $\mathbb{Z}_{M(G,k)}$ of the K-group (1.1).

For nontrivial outer automorphisms, a similar analysis was carried through in [13]. Here, the D-branes are parametrised by ω-twisted highest weight representations a of $\mathfrak{g}_k$ [14–17], and the NIM-rep coefficients are given by twisted fusion rules [17]. The twisted representations can be identified with representations of the invariant subalgebra $\bar{\mathfrak{g}}^\omega$ consisting of ω-invariant elements of $\bar{\mathfrak{g}}$, and we can view $\mathcal{B}_k^\omega$ as a subset of $\mathcal{P}_{k'}^+(\bar{\mathfrak{g}}^\omega)$, where $k' = k + h^\vee(\bar{\mathfrak{g}}) - h^\vee(\bar{\mathfrak{g}}^\omega)$. It was found that the charge q_a of $a \in \mathcal{B}_k^\omega$ is again (up to rescalings) given by the Weyl dimension[1] of the representation of $\bar{\mathfrak{g}}^\omega$, $q_a = \dim(a)$, and that the charge identities are only satisfied modulo $M = M(G,k)$. Thus each such class of twisted D-branes accounts for another summand $\mathbb{Z}_{M(G,k)}$ of the charge group. Since the number of automorphisms does not grow with the level, these constructions do not in general account for all the charges of (1.1); for the case of the A_n series, a proposal for the D-branes that may carry the remaining charges was made in [18, 19] (see also [20]).

The analysis of [13] was only done for all order-2 automorphisms of the classical Lie groups. There exist two 'exceptional' automorphisms, namely the order-3 automorphism of D_4 (triality), and the order-2 automorphism of E_6 (charge conjugation). These two cases are the subject of this paper. We will find again that with the charge assignment $q_a = \dim(a)$, the charge identities are satisfied modulo $M(G,k)$. Thus each such class of D-branes accounts for another summand of the charge group.[2]

[1]A similar proposal was made in [22] based on an analysis for large level.

[2]For the case of D_4, there are in total five 'twisted' classes of branes that are associated to ω, ω^2, C,

The plan of the paper is as follows. In the remainder of this section we shall explain the main steps in proving these results that are common to both cases. The details of the analysis for the case of the triality automorphism of D_4 (whose invariant subalgebra is G_2) is given in section 2. The corresponding analysis for the charge conjugation automorphism of E_6 (whose invariant subalgebra is F_4) is given in section 3.

1.1 Some notation and a sketch of the proof

We begin by briefly introducing some notation. The ω-twisted D-branes are characterised by the gluing conditions

$$\left(J^a_n + \omega(\bar{J}^a_{-n})\right) \|a\rangle\rangle = 0\,, \tag{1.5}$$

where J^a_n are the generators of $\mathfrak{g}$. Every boundary state can be written in terms of the ω-twisted Ishibashi states

$$\|a\rangle\rangle = \sum_{\mu\in\mathcal{E}^\omega_k} \psi_{a\mu}\,|\mu\rangle\rangle^\omega\,, \tag{1.6}$$

where $|\mu\rangle\rangle^\omega$ is the (up to normalisation) unique state satisfying (1.5) in the sector $\mathcal{H}_\mu\otimes\bar{\mathcal{H}}_{\mu^*}$. The sum in (1.6) runs over the so-called exponents that consist of the weights $\mu\in\mathcal{P}^+_k(\bar{\mathfrak{g}})$ that are invariant under ω. The NIM-rep coefficients are determined by the Verlinde-like formula

$$\mathcal{N}_{\lambda a}{}^b = \sum_{\mu\in\mathcal{E}^\omega_k} \frac{\psi^*_{b\mu}\,S_{\lambda\mu}\,\psi_{a\mu}}{S_{0\mu}}\,; \tag{1.7}$$

they define a non-negative integer matrix representation (NIM-rep) of the fusion rule algebra. (For a brief review of these matters see for example [13] and [21].)

It is clear on general grounds (see [13]) that for any charge assignment q_a for $a\in\mathcal{B}^\omega_k$, the charge identity (1.4) can at most be satisfied modulo $M(G,k)$. Our strategy will therefore be to construct a solution that solves (1.4) modulo $M(G,k)$. This solution is again given by $q_a = \dim(a)$. Furthermore, we can show that this solution of the charge equation is unique (up to trivial rescalings).

Our arguments will depend on the particularities of the two cases, but the general strategy is the same. The key observation of our analysis in both cases is a relation of the form

$$\mathcal{N}_{\lambda a}{}^b = \sum_{\gamma,i} \varphi_\lambda{}^\gamma\,\varepsilon_i\,N_{\gamma a}{}^{\rho_i(b)} \tag{1.8}$$

that expresses the NIM-rep coefficients $\mathcal{N}_{\lambda a}{}^b$ in terms of the fusion rules $N_{\gamma a}{}^{\rho_i(b)}$ of the affine algebra corresponding to $\bar{\mathfrak{g}}^\omega$. Here $\varphi_\lambda{}^\gamma$ is the branching coefficient which denotes how often the representation γ of $\bar{\mathfrak{g}}^\omega$ appears in the restriction of the representation λ to $\bar{\mathfrak{g}}^\omega$. The ρ_i are maps $\rho_i:\mathcal{B}^\omega_k\to\mathcal{P}^+_{k'}(\bar{\mathfrak{g}}^\omega)$ and ε_i is a sign attributed to the map ρ_i. Furthermore k' is defined as before, $k'=k+h^\vee(\bar{\mathfrak{g}})-h^\vee(\bar{\mathfrak{g}}^\omega)$. In the cases studied in [13] analogous

ωC and $\omega^2 C$, where C denotes charge conjugation. The corresponding NIM-reps are all closely related to the one discussed in this paper, or the charge conjugation NIM-rep discussed in [13] (see [17]). The arguments given here, together with the results of [13] therefore imply that these five twisted classes of D-branes account for five summands in (1.1).

JHEP05(2005)058

formulae for the NIM-rep coefficients were used for which the ρ_i could be expressed in terms of simple currents. In the current context where the invariant algebras are G_2 and F_4, such simple currents do not exist. Nevertheless it is possible to find such maps ρ_i (see (2.13) and (3.8) below for the specific formulae) that 'mimic' the action of the simple currents. The different maps ρ_i have disjoint images, and we can write

$$\mathcal{P}^+_{k'}(\bar{\mathfrak{g}}^\omega) = \bigcup_i \rho_i\left(\mathcal{B}^\omega_k\right) \;\; \cup \; \mathcal{R}_k \,, \tag{1.9}$$

where $\mathcal{R}_k$ denotes the remainder. The second key ingredient in our proof are the relations

$$\dim(\rho_i(a)) = \varepsilon_i \dim(a) \qquad a \in \mathcal{B}^\omega_k \tag{1.10}$$

$$\dim(b) = 0 \qquad b \in \mathcal{R}_k \,. \tag{1.11}$$

Both of these identities hold modulo $M(G^\omega, k')$. Finally we observe by explicit inspection of the above formulae for $M(G,k)$ that in the two cases of interest

$$M(G,k) = M(G^\omega, k') \,. \tag{1.12}$$

This then allows us to reduce the proof of the charge identities for the twisted D-branes of G to that of the untwisted D-branes of G^ω. In fact, the argument is simply

$$\begin{aligned}
\sum_{b\in\mathcal{B}^\omega_k} \mathcal{N}_{\lambda a}{}^{b} \dim_{G^\omega}(b) &= \sum_{b\in\mathcal{B}^\omega_k} \sum_i \sum_\gamma \varepsilon_i \, \varphi_\lambda{}^\gamma N_{\gamma a}{}^{\rho_i(b)} \dim_{G^\omega}(b) \\
&= \sum_{b\in\mathcal{B}^\omega_k} \sum_i \sum_\gamma \varphi_\lambda{}^\gamma N_{\gamma a}{}^{\rho_i(b)} \dim_{G^\omega}(\rho_i(b)) \qquad \bmod M(G,k) \\
&= \sum_\gamma \varphi_\lambda{}^\gamma \sum_{b\in\mathcal{P}^+_{k'}(\bar{\mathfrak{g}}^\omega)} N_{\gamma a}{}^{b} \dim_{G^\omega}(b) \qquad \bmod M(G,k) \\
&= \sum_\gamma \varphi_\lambda{}^\gamma \dim_{G^\omega}(\gamma) \dim_{G^\omega}(a) \qquad \bmod M(G,k) \\
&= \dim_G(\lambda) \dim_{G^\omega}(a) \,.
\end{aligned} \tag{1.13}$$

In the following two sections we shall give the details for how to define the maps ρ_i, and prove the various statements above. We shall also be able to show that our charge solution is unique up to trivial rescalings.

2. The D_4 case with triality

In the D_4 case the relevant automorphism ω is triality which maps the Dynkin labels $\mu = (\mu_0; \mu_1, \mu_2, \mu_3, \mu_4)$ to $(\mu_0; \mu_4, \mu_2, \mu_1, \mu_3)$. The set of exponents labelling the ω-twisted Ishibashi states is therefore

$$\mathcal{E}^\omega_k = \{(\mu_0; \mu_1, \mu_2, \mu_1, \mu_1) \in \mathbb{N}_0^5 \,|\, \mu_0 + 3\mu_1 + 2\mu_2 = k\} \,. \tag{2.1}$$

The ω-twisted boundary states are labelled by the level k integrable highest weights of the twisted Lie algebra $\mathfrak{g}^\omega = D_4^{(3)}$, which are $\mathcal{B}^\omega_k = \{(b_0; b_1, b_2) \in \mathbb{N}_0^3 \,|\, b_0 + 2b_1 + 3b_2 = k\}$. The states of lowest conformal weight of these representations form irreducible representations of the invariant subalgebra $\bar{\mathfrak{g}}^\omega = G_2$ with highest weights (b_1, b_2). For this reason we propose that the corresponding D-brane charge is the Weyl dimension of these irreducible representations, i.e.

$$q_b = \dim_{G_2}(b_1, b_2) = \dim_{G_2}(b)\,. \tag{2.2}$$

In this section we shall prove that (2.2) solves the charge constraint

$$\dim_{D_4}(\lambda)\, q_a = \sum_{b\in\mathcal{B}^\omega_k} \mathcal{N}_{\lambda a}{}^b\, q_b \tag{2.3}$$

modulo $M(G,k)$ and that this solution is unique (up to rescalings).

2.1 The solution

To show that (2.2) indeed solves the charge constraint (2.3) we trace the problem back to the case of untwisted branes in G_2. So we need to express 'twisted D_4 data' by 'untwisted G_2 data'. We first note, as already mentioned in section 1.1, that the integer M for G_2 at level $k+2$ equals the integer for D_4 at level k,

$$M(D_4,k) = M(G_2,k+2) = \frac{k+6}{\gcd(k+6,60)}\,. \tag{2.4}$$

As is also explained there, the key result (1.8) that we need to prove expresses the NIM-rep $\mathcal{N}$ of D_4 in terms of the fusion rules of G_2. The first step in providing such a relation is the identification of the D_4 ψ-matrix at level k with the (rescaled) S-matrix of G_2 at level $k+2$ (in the following we shall denote the S-matrix of G_2 by S' in order to distinguish it from the S-matrix of D_4)[3]

$$\psi_{b\mu} = \sqrt{3}\, S'_{b\tilde{\mu}}\,, \tag{2.5}$$

where $\tilde{\mu}$ is defined by

$$\mu \;\mapsto\; \tilde{\mu} = (\mu_0; 3\mu_1+2, \mu_2)\,. \tag{2.6}$$

Note that if $\mu \in \mathcal{E}^\omega_k$, then $\tilde{\mu} \in \mathcal{P}^+_{k+2}(G_2) \equiv \mathcal{P}_{k+2} = \{(\tilde{\mu}_1,\tilde{\mu}_2) \in \mathbb{N}_0^2 \,|\, \tilde{\mu}_1 + 2\tilde{\mu}_2 \le k+2\}$. The identity (2.5) can be proven as follows. Define $\kappa = k+6$ and $c(x) = \cos(\frac{2\pi x}{3\kappa})$. The ψ-matrix is given by (see [17]),

$$\begin{aligned}\psi_{b\mu} = \frac{2}{\kappa}\big(&c(pp'+2pq'+2qp'+qq') + c(2pp'+pq'+qp'-qq') +\\ &+ c(-pp'+pq'+qp'+2qq') - c(2pp'+pq'+qp'+2qq') -\\ &- c(pp'+2pq'-qp'+qq') - c(pp'-pq'+2qp'+qq')\big)\,,\end{aligned} \tag{2.7}$$

where $p = b_1+b_2+2$, $q = b_2+1$ and $p' = 3\mu_1+\mu_2+4$, $q' = \mu_2+1$. On the other hand, if we define $m = \lambda_1+\lambda_2+2$, $n = \lambda_2+1$, $m' = \nu_1+\nu_2+2$ and $n' = \nu_2+1$, then the S-matrix of G_2 at level $k+2$ is [23]

$$S'_{\lambda\nu} = \frac{-2}{\sqrt{3}\kappa}\big(c(2mm'+mn'+nm'+2nn') + c(-mm'-2mn'-nn'+nm') +$$

[3]This relation was already noted in [17].

$$
\begin{aligned}
&+ c(-mm' + mn' - 2nm' - nn') - c(-mm' - 2mn' - 2nm' - nn') - \\
&- c(2mm' + mn' + nm' - nn') - c(-mm' + mn' + nm' + 2nn')\big) \,. \qquad (2.8)
\end{aligned}
$$

For (2.8) we also use the abbreviated notation

$$
S'_{\lambda\nu} = \frac{-2}{\sqrt{3}\kappa} \{c(u_1) + c(u_2) + c(u_3) - c(u_4) - c(u_5) - c(u_6)\} \,. \qquad (2.9)
$$

By comparing (2.7) and (2.8) one then easily proves (2.5).

Next we observe from (1.7) that in order to obtain fusion matrices of G_2 we also need to express the quotient $\frac{S_{\lambda\mu}}{S_{0\mu}}$ in terms of G_2 S-matrices. The relevant relation is

$$
\frac{S_{\lambda\mu}}{S_{0\mu}} = \sum_\gamma {\varphi_\lambda}^\gamma \, \frac{S'_{\gamma\tilde{\mu}}}{S'_{0\tilde{\mu}}} \,. \qquad (2.10)
$$

Here, ${\varphi_\lambda}^\gamma$ denotes the $D_4 \supset G_2$ branching rules; the most important ones are

$$
(1,0,0,0) \;\to\; (1,0) \oplus (0,0) \qquad \text{and} \qquad (0,1,0,0) \;\to\; (0,1) \oplus (1,0) \oplus (1,0) \,. \qquad (2.11)
$$

The easiest way to prove (2.10) is to consider the explicit expressions for the fundamental representations of D_4.

Taking all of this together we can now write the D_4 NIM-rep as

$$
{\mathcal{N}_{\lambda a}}^b \;=\; \sum_{\mu \in \mathcal{E}^\omega_k} \psi_{a\mu} \, \psi^*_{b\mu} \, \frac{S_{\lambda\mu}}{S_{0\mu}} = 3 \sum_\gamma {\varphi_\lambda}^\gamma \sum_{\mu \in \mathcal{E}^\omega_k} S'_{a\tilde{\mu}} \, S'^*_{b\tilde{\mu}} \, \frac{S'_{\gamma\tilde{\mu}}}{S'_{0\tilde{\mu}}} \,. \qquad (2.12)
$$

Although this formula reminds one of the Verlinde formula, the last sum still does not give the G_2-fusion rules as the range of summation for $\tilde{\mu}$ is only a subset of $\mathcal{P}_{k+2}$.

To resolve this problem we introduce the affine mappings

$$
\begin{aligned}
\rho_0(b) &= (b_1, b_2) \\
\rho_1(b) &= (k - 2b_1 - 3b_2, 1 + b_1 + b_2) \\
\rho_2(b) &= (k + 1 - b_1 - 3b_2, b_2)
\end{aligned} \qquad (2.13)
$$

which map the set $\mathcal{B}^\omega_k$ of boundary states to disjoint subsets of $\mathcal{P}_{k+2} = \mathcal{P}^+_{k+2}(G_2)$. They have the crucial property

$$
S'_{\rho_0(b)\,\nu} + S'_{\rho_1(b)\,\nu} - S'_{\rho_2(b)\,\nu} = \begin{cases} 3\, S'_{b\,\nu} & \text{if } \nu_1 = 2 \bmod 3 \\ 0 & \text{otherwise,} \end{cases} \qquad (2.14)
$$

where $b \in \mathcal{B}^\omega_k$ and $\nu \in \mathcal{P}_{k+2}$. This follows from the fact that the left hand side can be written as

$$
\begin{aligned}
\frac{-\sqrt{3}\kappa}{2} \left(S'_{b\,\nu} + S'_{\rho_1(b)\,\nu} - S'_{\rho_2(b)\,\nu} \right) \;&=\; \big(1 + \cos(v_1) + \cos(v_2)\big)\big(c(u_1) - c(u_4)\big) + \\
&+\; \big(1 + \cos(v_1) + \cos(v_3)\big)\big(c(u_2) - c(u_6)\big) + \\
&+\; \big(1 + \cos(v_2) + \cos(v_3)\big)\big(c(u_3) - c(u_5)\big) +
\end{aligned}
$$

$$
\begin{aligned}
&+\big(\sin(v_1)+\sin(v_2)\big)\big(s(u_1)+s(u_4)\big)-\\
&-\big(\sin(v_1)-\sin(v_3)\big)\big(s(u_2)+s(u_6)\big)-\\
&-\big(\sin(v_2)+\sin(v_3)\big)\big(s(u_3)+s(u_5)\big)\,,
\end{aligned}
\tag{2.15}
$$

where $v_1=\frac{2}{3}\pi(\nu_1+3\nu_2+4)$, $v_2=\frac{2}{3}\pi(2\nu_1+3\nu_2+5)$, $v_3=\frac{2}{3}\pi(\nu_1+1)$ and $s(x)=\sin\left(\frac{2\pi x}{3\kappa}\right)$. (The u_i are defined as in (2.9).) This is easily seen to agree with the right hand side of (2.14).

Let $\rho(\mathcal{B}_k^\omega)=\rho_0(\mathcal{B}_k^\omega)\cup\rho_1(\mathcal{B}_k^\omega)\cup\rho_2(\mathcal{B}_k^\omega)$ and $\mathcal{R}_k=\mathcal{P}_{k+2}\setminus\rho(\mathcal{B}_k^\omega)$. The special elements $\nu\in\mathcal{P}_{k+2}$ in (2.14) which satisfy $\nu_1=2 \bmod 3$ are precisely the images $\nu=\tilde{\mu}$ under (2.6) of a suitable element μ of $\mathcal{E}_k^\omega$. The key relation (2.14), together with (2.12), therefore implies that the D_4 NIM-rep can be written as a sum of G_2 fusion matrices,

$$
\begin{aligned}
\mathcal{N}_{\lambda a}{}^{b} &= \sum_\gamma \varphi_\lambda{}^\gamma \sum_{\mu\in\mathcal{P}_{k+2}} S'_{a\mu}\frac{S'_{\gamma\mu}}{S'_{0\mu}}\left(S'^{*}_{\rho_0(b)\,\mu}+S'^{*}_{\rho_1(b)\,\mu}-S'^{*}_{\rho_2(b)\,\mu}\right)\\
&=\sum_\gamma \varphi_\lambda{}^\gamma\left(N_{\gamma a}{}^{\rho_0(b)}+N_{\gamma a}{}^{\rho_1(b)}-N_{\gamma a}{}^{\rho_2(b)}\right)\\
&=\sum_{i=0}^{2}\sum_\gamma \varepsilon_i\,\varphi_\lambda{}^\gamma N_{\gamma a}{}^{\rho_i(b)}\,,
\end{aligned}
\tag{2.16}
$$

where N_γ denote G_2 fusion matrices at level $k+2$ and ε_i accounts for the signs. (Explicitly $\varepsilon_0=\varepsilon_1=+1$ and $\varepsilon_2=-1$.) This is the relation (1.8) we proposed in section 1.1. Note that (2.16) is valid for all highest weights λ of D_4, not only for the ones appearing in $\mathcal{P}_k^+(D_4)$. In fact we can continue $\mathcal{N}_\lambda$ and N_γ outside of the usual domain by rewriting the ratios of S-matrices appearing in (2.12) as characters of the finite Lie algebras.

According to the argument given in section 1.1, there is only one further ingredient we need to show. This concerns the behaviour of the G_2-Weyl dimensions under the maps ρ_i, and is summarised in the relations (1.10) and (1.11). Thus we need to prove that

$$
\dim_{G_2}(\rho_i(b))=\varepsilon_i\,\dim_{G_2}(b)\qquad \bmod M(G_2,k+2)\,,
\tag{2.17}
$$

and that any element $r\in\mathcal{R}_k$ satisfies

$$
\dim_{G_2}(r)=0\qquad \bmod M(G_2,k+2)\,.
\tag{2.18}
$$

The dimension of a G_2 representation (b_1,b_2) is given by

$$
\dim_{G_2}(b_1,b_2)=\frac{1}{120}(b_1+1)(b_2+1)(b_1+b_2+2)(b_1+2b_2+3)(b_1+3b_2+4)(2b_1+3b_2+5)\ .
$$

To prove (2.17) we find by explicit computation that

$$
\dim_{G_2}(\rho_i(b))=\varepsilon_i\,\dim_{G_2}(b)+\frac{M(G_2,k+2)}{F}\,p^5(b)\,,
\tag{2.19}
$$

where $p^5(b)$ denotes a k and ρ_i-dependent polynomial of order 5 in the variables b_1,b_2 with integer coefficients, and $F=\frac{120}{\gcd(k+6,60)}$. Thus it remains to show that $\frac{p^5}{F}$ is an integer. If 8 does not divide $k+6$, then $M(G_2,k+2)$ and F are coprime. Since $\frac{M(G_2,k+2)}{F}p^5$ is an integer, $\frac{p^5}{F}$ has to be an integer as well and we are done. If 8 is a divisor of $k+6$, then F and $M(G_2, k+2)$ have greatest common divisor 2. The result then follows provided that p^5 is even, which is easily verified.

To show (2.18) we first have to identify the elements of $\mathcal{R}_k$. It is convenient to write this set as the (disjoint) union of the two subsets $\mathcal{R}_k^1$ and $\mathcal{R}_k^2$. The first of them is defined by

$$\mathcal{R}_k^1 = \{(b_1, b_2) \in \mathcal{P}_{k+2} \,|\, (b_1, b_2) = (k+2-3j, j),\ j \in \mathbb{N}_0\}\ . \tag{2.20}$$

The set $\mathcal{R}_k^2 \equiv \mathcal{R}_k \backslash \mathcal{R}_k^1$ depends in a more complicated manner on k. To describe it explicitly we therefore distinguish the three cases:

- $k = 0 \bmod 3$
$$\mathcal{R}_k^2 = \left\{ \left(2+3j, \frac{k}{3} - 1 - 2j\right) \in \mathcal{P}_{k+2},\ j \in \mathbb{N}_0 \right\}$$
- $k = 1 \bmod 3$
$$\mathcal{R}_k^2 = \left\{ \left(0, \frac{(k+2)}{3}\right) \right\} \cup \left\{ \left(1+3j, \frac{(k-1)}{3} - 2j\right) \in \mathcal{P}_{k+2},\ j \in \mathbb{N}_0 \right\}$$
- $k = 2 \bmod 3$
$$\mathcal{R}_k^2 = \left\{ \left(0, \frac{(k+1)}{3}\right) \right\} \cup \left\{ \left(3+3j, \frac{(k+1)}{3} - 2(j+1)\right) \in \mathcal{P}_{k+2},\ j \in \mathbb{N}_0 \right\}.$$

For any $r \in \mathcal{R}_k$ one then easily checks that

$$\dim_{G_2}(r) = \frac{M}{F} p^5(j) \tag{2.21}$$

with some polynomial p^5 in j of order 5. One finds that the polynomials $p^5(j)$ are even whenever 8 divides $k+6$. Using the same arguments as above, this then finishes the proof of (2.18). It remains to check that we have identified the complete set $\mathcal{R}_k$ correctly. Because $\rho_i(\mathcal{B}_k^\omega) \cap \rho_j(\mathcal{B}_k^\omega) = \emptyset$ for $i \neq j$ we have $|\rho(\mathcal{B}_k^\omega)| = 3\,|\mathcal{B}_k^\omega|$. In order to see that $\mathcal{P}_{k+2} = \rho(\mathcal{B}_k^\omega) \cup \mathcal{R}_k$, it is therefore sufficient to count the number of elements of the different sets. One easily finds

$$|\mathcal{P}_{k+2}| = \begin{cases} \frac{1}{4}(k+4)^2 & k \text{ even} \\ \frac{1}{4}(k+3)(k+5) & k \text{ odd} \end{cases}$$

as well as

$$|\mathcal{R}_k| = \begin{cases} \frac{1}{2}(k+2) & k = 0 \quad \bmod 6 \\ \frac{1}{2}(k+4) & k = 2,4 \ \bmod 6 \\ \frac{1}{2}(k+3) & k = 3 \quad \bmod 6 \\ \frac{1}{2}(k+5) & k = 1,5 \ \bmod 6 \end{cases} \qquad |\mathcal{B}_k^\omega| = \begin{cases} \frac{1}{12}k^2 + \frac{1}{2}k + 1 & k = 0 \quad \bmod 6 \\ \frac{1}{12}(k+2)(k+4) & k = 2,4 \ \bmod 6 \\ \frac{1}{12}(k+3)^2 & k = 3 \quad \bmod 6 \\ \frac{1}{12}(k+1)(k+5) & k = 1,5 \ \bmod 6 \end{cases}$$

Using these formulae it is then easy to show that $|\mathcal{P}_{k+2}| = |\rho(\mathcal{B}_k^\omega)| + |\mathcal{R}_k|$. This completes the proof.

2.2 Uniqueness

It remains to prove that the solution we found is unique up to an overall rescaling of the charge. To this end we show that any solution of the charge constraint modulo some integer M' satisfies the relation

$$q_a = \dim(a)\, q_0 \quad \bmod M' \,, \tag{2.22}$$

and thus is obtained from our solution (2.2) by scaling with the factor q_0.

To prove (2.22) we first want to show that any G_2 representation a can be obtained as restriction of a linear combination of D_4 representations λ_j with integer coefficients z_j. We explicitly allow negative multiplicities and write formally

$$a = \bigoplus_j z_j \lambda_j \big|_{G_2} .$$

Obviously it is sufficient to prove this for the fundamental representations. Looking at the branching rules (2.11) we see that the representation $(1,0)$ appears in the decomposition of $(1,0,0,0)$, so we can write

$$(1,0) = ((1,0,0,0) - (0,0,0,0))\,\big|_{G_2} .$$

Similarly, we can express $(0,1)$ as a restriction because it appears exactly once in the decomposition of $(0,1,0,0)$ together only with $(1,0)$ (see (2.11)).

Now consider a boundary state labelled by a. We can use (2.16) to write[4]

$$\begin{aligned} \dim_{G_2}(a)\, q_0 &= \sum_j z_j \, \dim_{D_4}(\lambda_j)\, q_0 \\ &= \sum_{j,b} z_j \, \mathcal{N}_{\lambda_j 0}{}^{b} q_b \qquad \bmod M' \\ &= \sum_{i,j,\gamma,b} z_j \, \varepsilon_i \, \varphi_{\lambda_j}{}^{\gamma} \, N_{\gamma 0}{}^{\rho_i(b)} \, q_b \\ &= \sum_{i,b} \varepsilon_i \, N_{a0}{}^{\rho_i(b)} \, q_b \\ &= q_a \,. \end{aligned} \tag{2.23}$$

In the last step we used the fact that $\rho_i(\mathcal{B}_k^\omega)$ and $\mathcal{B}_k^\omega$ are disjoint for $i \neq 0$, so that only $i = 0$ contributes. This concludes the proof of (2.22).

3. The E_6 case with charge conjugation

The analysis for the case of E_6 is fairly similar to the D_4 case discussed in the previous section, and we shall therefore be somewhat briefer. For E_6 the invariant subalgebra under charge conjugation is $\bar{\mathfrak{g}}^\omega = F_4$. Again we have the identity

$$M(E_6, k) = M(F_4, k+3) = \frac{k+12}{\gcd(k+12, 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11)} . \tag{3.1}$$

[4] Note that the charge constraint (2.3) as well as the expression (2.16) for the NIM-rep is valid also for highest weights λ which are not in $\mathcal{P}_k^+(D_4)$.

JHEP05(2005)058

As before we therefore expect that

$$\dim_{E_6}(\lambda)\, q_a = \sum_{b\in\mathcal{B}_k^\omega} \mathcal{N}_{\lambda a}{}^b\, q_b \qquad \mod M(F_4, k+3)\,, \tag{3.2}$$

where

$$q_b = \dim_{F_4}(b)\,. \tag{3.3}$$

The order 2 automorphism ω of E_6 maps the Dynkin labels $(\mu_0;\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6)$ to $(\mu_0;\mu_5,\mu_4,\mu_3,\mu_2,\mu_1,\mu_6)$, and thus the set of exponents is

$$\mathcal{E}_k^\omega = \big\{(\mu_0;\mu_1,\mu_2,\mu_3,\mu_2,\mu_1,\mu_6) \in \mathbb{N}_0^7 \,|\, \mu_0 + 2\mu_1 + 4\mu_2 + 3\mu_3 + 2\mu_6 = k\big\}\,. \tag{3.4}$$

The twisted algebra here is $E_6^{(2)}$. The set of boundary states at level k is explicitly given by $\mathcal{B}_k^\omega = \{(b_0;b_1,b_2,b_3,b_4) \in \mathbb{N}_0^5 \,|\, b_0 + 2b_1 + 3b_2 + 4b_3 + 2b_4 = k\}$. As in the last section, it is possible to identify the ψ-matrix of E_6 at level k with the S-matrix of F_4 at level $k+3$ (see also [17]),

$$\psi_{b\mu} = 2\, S'_{b\tilde{\mu}}\,, \tag{3.5}$$

where $\tilde{\mu}$ is now defined by

$$\mu \;\mapsto\; \tilde{\mu} = (\mu_0; 2\mu_1+1, 2\mu_2+1, \mu_3, \mu_6)\,. \tag{3.6}$$

As before we observe that if $\mu \in \mathcal{E}_k^\omega$, then $\tilde{\mu} \in \mathcal{P}_{k+3}^+(F_4) \equiv \mathcal{P}_{k+3}$, where the latter is explicitly defined as $\mathcal{P}_{k+3} = \{(\tilde{\mu}_1,\tilde{\mu}_2,\tilde{\mu}_3,\tilde{\mu}_4) \in \mathbb{N}_0^4 \,|\, \tilde{\mu}_1 + 2\tilde{\mu}_2 + 3\tilde{\mu}_3 + 2\tilde{\mu}_4 \le k+3\}$. Furthermore, we can express ratios of S-matrices of E_6 by those of F_4,

$$\frac{S_{\lambda\mu}}{S_{0\mu}} = \sum_\gamma \varphi_\lambda{}^\gamma\, \frac{S'_{\gamma\tilde{\mu}}}{S'_{0\tilde{\mu}}}\,.$$

Here S denotes the E_6 S-matrix at level k, S' is the F_4 S-matrix at level $k+3$, and $\varphi_\lambda{}^\gamma$ describes the branching $E_6 \supset F_4$; for the six fundamental representations of E_6 the branching rules are

$$\begin{aligned}
(1,0,0,0,0,0) &\simeq (0,0,0,0,1,0) \to (1,0,0,0) \oplus (0,0,0,0)\\
(0,1,0,0,0,0) &\simeq (0,0,0,1,0,0) \to (0,1,0,0) \oplus (0,0,0,1) \oplus (1,0,0,0)\\
(0,0,1,0,0,0) &\to (0,0,1,0) \oplus (1,0,0,1) \oplus 2\cdot(0,1,0,0) \oplus (0,0,0,1)\\
(0,0,0,0,0,1) &\to (0,0,0,1) \oplus (1,0,0,0)\,.
\end{aligned} \tag{3.7}$$

The relevant affine mappings are in this case

$$\begin{aligned}
\rho_0(b) &= (b_1, b_2, b_3, b_4)\\
\rho_1(b) &= (k - 2b_1 - 3b_2 - 4b_3 - 2b_4, 1 + b_1 + b_2, b_3, b_4)\\
\rho_2(b) &= (k + 1 - b_1 - 3b_2 - 4b_3 - 2b_4, b_2, b_3, b_4)\\
\rho_3(b) &= (k - 2b_1 - 3b_2 - 4b_3 - 2b_4, b_1, b_2 + b_3 + 1, b_4)\,,
\end{aligned} \tag{3.8}$$

which map boundary states at level k to dominant weights of F_4 at level $k+3$, i.e. to elements of $\mathcal{P}_{k+3}$. There is a similar identity to (2.14) for the S-matrices

$$S'_{\rho_0(b)\,\nu} + S'_{\rho_1(b)\,\nu} - S'_{\rho_2(b)\,\nu} - S'_{\rho_3(b)\,\nu} \;=\; \begin{cases} 4\,S'_{b\,\nu} & \text{if } \nu_1 = \nu_2 = 1 \bmod 2 \\ 0 & \text{otherwise,} \end{cases} \tag{3.9}$$

where $b \in \mathcal{B}^\omega_k$ and $\nu \in \mathcal{P}_{k+3}$. Again, the elements which satisfy $\nu_1 = \nu_2 = 1 \bmod 2$ are precisely the images $\nu = \tilde{\mu}$ of an element μ of $\mathcal{E}^\omega_k$ under the mapping (3.6). By $\rho(\mathcal{B}^\omega_k)$ we denote the union of the images of $\mathcal{B}^\omega_k$ under the maps ρ_i, $\rho(\mathcal{B}^\omega_k) = \bigcup_{i=0}^3 \rho_i(\mathcal{B}^\omega_k)$. The elements of $\mathcal{P}_{k+3}$ which are not reached by the maps form the set $\mathcal{R}_k = \mathcal{P}_{k+3} \setminus \rho(\mathcal{B}^\omega_k)$.

Using essentially the same arguments as for the case of D_4 discussed in the last section, we can then show that the E_6 NIM-rep can be expressed in terms of F_4 fusion matrices as

$$\mathcal{N}_{\lambda a}{}^{b} = \sum_\gamma \varphi_\lambda{}^\gamma \left(N_{\gamma a}{}^{\rho_0(b)} + N_{\gamma a}{}^{\rho_1(b)} - N_{\gamma a}{}^{\rho_2(b)} - N_{\gamma a}{}^{\rho_3(b)} \right) = \sum_{i=0}^{3} \sum_\gamma \varepsilon_i\, \varphi_\lambda{}^\gamma N_{\gamma a}{}^{\rho_i(b)}\,, \tag{3.10}$$

where the ε_i account for the signs. (Explicitly, $\varepsilon_1 = \varepsilon_2 = +1$ and $\varepsilon_3 = \varepsilon_4 = -1$.) Following the argument of section 1.1, it thus only remains to show that

$$\dim_{F_4}(\rho_i(b)) = \varepsilon_i \dim_{F_4}(b) \qquad \bmod M(F_4, k+3)\,, \tag{3.11}$$

and for all $r \in \mathcal{R}_k$

$$\dim_{F_4}(r) = 0 \qquad \bmod M(F_4, k+3)\,. \tag{3.12}$$

To prove equation (3.11) we note that

$$dim_{F_4}(\rho_i(b)) = \varepsilon_i \dim_{F_4}(b) + \frac{M(F_4, k+3)}{F}\, p^{23}(b)\,, \tag{3.13}$$

where

$$F = \frac{2^{15} \cdot 3^7 \cdot 5^4 \cdot 7^2 \cdot 11}{\gcd(k+12, 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11)} \tag{3.14}$$

and p^{23} is a k and ρ_i-dependent polynomial (with integer coefficients) of degree 23 in the labels b_i. Now $M(F_4, k+3)$ and F are coprime whenever 2^4, 3^3, 5^2 and 7^2 do not divide $k+12$; in this case (3.11) is proven as before. Otherwise the analysis is more involved and many cases would have to be distinguished. We have not attempted to analyse all of them in detail, but we have performed a numerical check up to fairly high levels. This seems satisfactory, given that the identities for $M(E_6, k)$ and $M(F_4, k)$ have also only be determined numerically.

Finally, we need to show the identity (3.12). This requires a good description of the set $\mathcal{R}_k$. Here it is convenient to write it as the union of four disjoint subsets which are defined by

$$\mathcal{R}^1_k \;=\; \begin{cases} \{b \in \mathcal{P}_{k+3} \,|\, b = (1+2j_1, j_2, j_3, (k+2)/2 - j_1 - j_2 - 2j_3)\} & k \text{ even} \\ \{b \in \mathcal{P}_{k+3} \,|\, b = (2j_1, j_2, j_3, (k+3)/2 - j_1 - j_2 - 2j_3)\} & k \text{ odd} \end{cases}$$

$$\mathcal{R}_k^2 = \begin{cases} \{b \in \mathcal{P}_{k+3} \,|\, b = (2j_1, 2j_2, j_3, (k+2)/2 - j_1 - 3j_2 - 2j_3)\} & k \text{ even} \\ \{b \in \mathcal{P}_{k+3} \,|\, b = (1+2j_1, 2j_2, j_3, (k+1)/2 - j_1 - 3j_2 - 2j_3)\} & k \text{ odd} \end{cases}$$

$$\mathcal{R}_k^3 = \begin{cases} \{b \in \mathcal{P}_{k+3} \,|\, b = (1+2j_1, 1+2j_2, j_3, (k-2)/2 - j_1 - 3j_2 - 2j_3)\} & k \text{ even} \\ \{b \in \mathcal{P}_{k+3} \,|\, b = (2j_1, 1+2j_2, j_3, (k-1)/2 - j_1 - 3j_2 - 2j_3)\} & k \text{ odd} \end{cases}$$

$$\mathcal{R}_k^4 = \begin{cases} \{b \in \mathcal{P}_{k+3} \,|\, b = (j_1, 1+2j_2, j_3, (k-2)/2 - j_1 - 3j_2 - 2j_3)\} & k \text{ even} \\ \{b \in \mathcal{P}_{k+3} \,|\, b = (j_1, 2j_2, j_3, (k+1)/2 - j_1 - 3j_2 - 2j_3)\} & k \text{ odd,} \end{cases}$$

where $(j_1, j_2, j_3) \in \mathbb{N}_0^3$. The same arguments as before show that the elements r in these sets satisfy $\dim_{F_4}(r) = 0 \mod M(F_4, k+3)$. Again, this is proven only if $k+12$ is not divisible by $2^4, 3^3, 5^2$ or 7^2; for the other levels we have only performed numerical checks.

Finally, by counting the elements of the different sets we can confirm (as before) that we have correctly identified the set $\mathcal{R}_k$. This completes the proof for the case of E_6.

3.1 Uniqueness

The proof of uniqueness is analogous to the D_4 case. It only remains to show that all fundamental representations of F_4 can be obtained as restrictions of E_6 representations. From the branching rules (3.7) we see immediately that this is true for $(1,0,0,0)$, $(0,0,0,1)$ and $(0,1,0,0)$. The remaining fundamental representation $(0,0,1,0)$ appears in the decomposition of $(0,0,1,0,0,0)$, but it comes together with $(1,0,0,1)$. The latter representation can be obtained from the other fundamentals by the F_4 tensor product

$$(1,0,0,0) \otimes (0,0,0,1) \;\rightarrow\; (1,0,0,1) \oplus (1,0,0,0) \oplus (0,1,0,0)\,.$$

Hence, also $(0,0,1,0)$ can be written in terms of the restriction of D_4-representations.

Note added: while we were in the process of writing up this paper we became aware of [24] which contains closely related work.

Acknowledgments

This research has been partially supported by the Swiss National Science Foundation and the Marie Curie network ‘Constituents, Fundamental Forces and Symmetries of the Universe’ (MRTN-CT-2004-005104). The work of S.F. was supported by the Max Planck Society and the Max Planck Institute for Gravitational Physics in Golm. We thank Terry Gannon for useful communications. This paper is largely based on the Diploma thesis of T.M. [25].

References

[1] R. Minasian and G.W. Moore, *K-theory and Ramond-Ramond charge*, *JHEP* **11** (1997) 002 [hep-th/9710230].

[2] E. Witten, *D-branes and K-theory*, *JHEP* **12** (1998) 019 [hep-th/9810188].

[3] G.W. Moore, *K-theory from a physical perspective*, hep-th/0304018.

[4] A. Kapustin, *D-branes in a topologically nontrivial B-field*, *Adv. Theor. Math. Phys.* **4** (2000) 127 [hep-th/9909089].

[5] P. Bouwknegt and V. Mathai, *D-branes, B-fields and twisted K-theory*, *JHEP* **03** (2000) 007 [hep-th/0002023].

[6] V. Braun, *Twisted K-theory of Lie groups*, *JHEP* **03** (2004) 029 [hep-th/0305178].

[7] D. Freed, M. Hopkins and C. Teleman, *Twisted K-theory and loop group representations*, math.at/0312155 and unpublished

[8] C. Douglas, *On the twisted K-homology of simple Lie groups*, math.at/0402082.

[9] S. Fredenhagen and V. Schomerus, *Branes on group manifolds, gluon condensates and twisted K-theory*, *JHEP* **04** (2001) 007 [hep-th/0012164].

[10] J.M. Maldacena, G.W. Moore and N. Seiberg, *D-brane instantons and K-theory charges*, *JHEP* **11** (2001) 062 [hep-th/0108100].

[11] P. Bouwknegt, P. Dawson and D. Ridout, *D-branes on group manifolds and fusion rings*, *JHEP* **12** (2002) 065 [hep-th/0210302].

[12] A. Alekseev and V. Schomerus, *RR charges of D2-branes in the WZW model*, hep-th/0007096.

[13] M.R. Gaberdiel and T. Gannon, *The charges of a twisted brane*, *JHEP* **01** (2004) 018 [hep-th/0311242].

[14] L. Birke, J. Fuchs and C. Schweigert, *Symmetry breaking boundary conditions and WZW orbifolds*, *Adv. Theor. Math. Phys.* **3** (1999) 671 [hep-th/9905038].

[15] J. Fuchs and C. Schweigert, *Solitonic sectors, α-induction and symmetry breaking boundaries*, *Phys. Lett.* **B 490** (2000) 163 [hep-th/0006181].

[16] V.B. Petkova and J.B. Zuber, *Boundary conditions in charge conjugate* $\mathrm{SL}(N)$ *WZW theories*, hep-th/0201239.

[17] M.R. Gaberdiel and T. Gannon, *Boundary states for WZW models*, *Nucl. Phys.* **B 639** (2002) 471 [hep-th/0202067].

[18] M.R. Gaberdiel, T. Gannon and D. Roggenkamp, *The D-branes of* $\mathrm{SU}(N)$, *JHEP* **07** (2004) 015 [hep-th/0403271].

[19] M.R. Gaberdiel, T. Gannon and D. Roggenkamp, *The coset D-branes of* $\mathrm{SU}(N)$, *JHEP* **10** (2004) 047 [hep-th/0404112].

[20] J.M. Maldacena, G.W. Moore and N. Seiberg, *D-brane instantons and K-theory charges*, *JHEP* **11** (2001) 062 [hep-th/0108100].

[21] R.E. Behrend, P.A. Pearce, V.B. Petkova and J.-B. Zuber, *Boundary conditions in rational conformal field theories*, *Nucl. Phys.* **B 570** (2000) 525 [hep-th/9908036].

[22] A.Y. Alekseev, S. Fredenhagen, T. Quella and V. Schomerus, *Non-commutative gauge theory of twisted D-branes*, *Nucl. Phys.* **B 646** (2002) 127 [hep-th/0205123].

[23] T. Gannon, *Algorithms for affine Kac-Moody algebras*, hep-th/0106123.

[24] M. Vasudevan, *Charges of exceptionally twisted branes*, hep-th/0504006.

[25] T. Mettler, *D-brane charges in WZW models*, Diploma thesis ETH Zurich, March 2005.